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Statistical-Mechanical Entropy of Garfinkle-Horowitz-Strominger Black Hole

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Abstract Taking WKB approximation to solve the scalar field equation in the Garfinkle-Horowitz-Strominger (GHS) black hole spacetime, we can get the classical momenta. Substituting the classical momenta into state density equation corrected by the generalized uncertainty principle, we will obtain the number of quantum states with energy less than ω. It is convergent in the neighborhood of the horizon. Then, it is used to calculate the statistical-mechanical entropy of the scalar field in the GHS black hole spacetime. The calculation shows that the entropy is proportional to the horizon area.

Keywords Generalized uncertainty principle · State density · Statistical-mechanical entropy · GHS black hole

1 Introduction

Bekenstein and Hawking [1, 2] found that the black hole entropy is proportional to the area of the event horizon by comparing black hole physics with thermodynamics. This is one of the most profound discoveries in the modern physics. Entropy is a statistical-mechanical concept, but the study of General Relativity shows that black hole has no hair. So the statistical origin of black hole entropy become an important question in theoretical physics. A progress has been made by 't Hooft [3], whose brick wall model (BWM) is extensively used to calculate the entropy in a variety of black holes. In this model the Bekenstein-Hawking entropy is identified with the statistical-mechanical entropy arising from a thermal bath of quantum fields propagating outside the event horizon. Because the number of quantum states is divergent at the horizon, the entropy is divergent unless introducing a cutoff ε [4–14]. Recently, people found that the generalized uncertainty principle would correct the

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state density. Ref. [15] gives following corrected state density to leading order in the Planck length

$$dn = \frac{d^3 x d^3 p}{(2\pi)^3 (1+\lambda p^2)^3}. \tag{1}$$

Refs. [16, 17] give the corrected state density to all orders in the Planck length

$$dn = e^{-\lambda p^2} \frac{d^3 x d^3 p}{(2\pi)^3}, \tag{2}$$

where λ is a constant characterizing the correction to Heisenberg uncertainty principle by gravitation and is equal to Planck area in order of magnitude, and $p^2 = p^i p_i$.

Up to now, a series of papers have studied black hole entropy by using (1) [18–21]. In this paper, we calculate the statistical-mechanical entropy of GHS black hole by using the corrected state density (2).

GHS black hole is a static, spherically symmetric charged black hole solution to low-energy string theory, which is given by Refs. [22, 23]. Taking the natural unit $(G = c = \hbar = 1)$, we can write its metric as follows

$$ds^2 = -\left(1 - \frac{r_+}{r}\right)\left(1 - \frac{r_-}{r}\right)^{\frac{1-a^2}{1+a^2}} dt^2 + \left(1 - \frac{r_+}{r}\right)^{-1}\left(1 - \frac{r_-}{r}\right)^{-\frac{1-a^2}{1+a^2}} dr^2$$
$$+ r^2 \left(1 - \frac{r_-}{r}\right)^{\frac{2a^2}{1+a^2}} \left(d\theta^2 + \sin^2\theta d\varphi^2\right), \tag{3}$$

where a is an arbitrary parameter governing the force of the coupling between the dilaton and the Maxwell field, r_+ and r_- are related to the physical mass and charge by

$$M = \frac{r_+}{2} + \left(\frac{1-a^2}{1+a^2}\right)\frac{r_-}{2}, \tag{4}$$

$$Q = \left(\frac{r_+ r_-}{1+a^2}\right)^{\frac{1}{2}} \tag{5}$$

and $r = r_+$ is its regular event horizon for any value of a, $r = r_-$ is its inner horizon which is singular for nonzero value of a.

2 Statistical-Mechanical Entropy of GHS Black Hole

Massless scalar field is described by the massless Klein-Gorden equation

$$\frac{1}{\sqrt{-g}} \frac{\partial}{\partial x^\mu}\left[\sqrt{-g} g^{\mu\nu} \frac{\partial \Phi}{\partial x^\nu}\right] = 0. \tag{6}$$

Substituting metric (3) into (6), we obtain

$$g^{tt} \frac{\partial^2 \Phi}{\partial t^2} + g^{rr} \frac{\partial^2 \Phi}{\partial r^2} + g^{\theta\theta} \frac{\partial^2 \Phi}{\partial \theta^2} + g^{\varphi\varphi} \frac{\partial^2 \Phi}{\partial \varphi^2}$$
$$+ \frac{1}{\sqrt{-g}} \frac{\partial}{\partial r}\left(\sqrt{-g} g^{rr}\right) \frac{\partial \Phi}{\partial r} + \frac{1}{\sqrt{-g}} \frac{\partial}{\partial \theta}\left(\sqrt{-g} g^{\theta\theta}\right) \frac{\partial \Phi}{\partial \theta} = 0. \tag{7}$$

Using Wentzel-Kramers-Brillouin (WKB) approximation with

$$\Phi = e^{-i\omega t} e^{iS(r,\theta,\varphi)}. \tag{8}$$

We find

$$-\omega^2 g^{tt} - g^{rr} p_r^2 - g^{\theta\theta} p_\theta^2 - g^{\varphi\varphi} p_\varphi^2 = 0, \tag{9}$$

where p_r, p_θ, p_φ are classical momenta. We can easily obtain following equations from (9)

$$p_r^2 = \frac{1}{g^{rr}}(-\omega^2 g^{tt} - g^{\theta\theta} p_\theta^2 - g^{\varphi\varphi} p_\varphi^2), \tag{10}$$

$$\begin{aligned} p^2 &= g^{rr} p_r^2 + g^{\theta\theta} p_\theta^2 + g^{\varphi\varphi} p_\varphi^2 \\ &= -\omega^2 g^{tt}. \end{aligned} \tag{11}$$

Using (2) the number of quantum states less than energy ω is given by

$$\begin{aligned} g(\omega) &= \int \frac{1}{(2\pi)^3} e^{-\lambda p^2} d^3x d^3p \\ &= \int \frac{1}{(2\pi)^3} e^{-\lambda p^2} dr d\theta d\varphi dp_r dp_\theta dp_\varphi \\ &= \frac{2}{(2\pi)^3} \int e^{-\lambda p^2} \frac{1}{\sqrt{g^{rr}}} \left(-\omega^2 g^{tt} - g^{\theta\theta} p_\theta^2 - g^{\varphi\varphi} p_\varphi^2\right)^{\frac{1}{2}} dr d\theta d\varphi dp_\theta dp_\varphi \\ &= \frac{2}{3\pi} \int \omega^3 e^{-\lambda\omega^2 (1-\frac{r_+}{r})^{-1} (1-\frac{r_-}{r})^{-\frac{1-a^2}{1+a^2}}} r^2 \left(1 - \frac{r_+}{r}\right)^{-2} \left(1 - \frac{r_-}{r}\right)^{-\frac{2(1-a^2)}{1+a^2} + \frac{2a^2}{1+a^2}} dr. \end{aligned} \tag{12}$$

We let

$$\left(1 - \frac{r_+}{r}\right)\left(1 - \frac{r_-}{r}\right)^{\frac{1-a^2}{1+a^2}} = f(r), \tag{13}$$

then (12) can be written as follows

$$g(\omega) = \frac{2}{3\pi} \int \omega^3 e^{-\frac{\lambda\omega^2}{f}} \frac{1}{f^2} r^2 \left(1 - \frac{r_-}{r}\right)^{\frac{2a}{1+a^2}} dr. \tag{14}$$

The above integral interval for r is $[r_+, r_+ + \varepsilon]$, where ε correspond with the thin layer inherent thickness $\sqrt{\lambda}$. It satisfies the following identity [21]

$$\sqrt{\lambda} = \int_{r_+}^{r_+ + \varepsilon} \sqrt{g_{rr}} dr \approx \sqrt{\frac{2\varepsilon}{\kappa}}, \tag{15}$$

where κ is the surface gravity of the black hole [24]. Additionally, $f(r_+ + \varepsilon)$ can be expanded to the ε at the vicinity of horizon r_+

$$f(r_+ + \varepsilon) \approx f(r_+) + f'(r_+)\varepsilon = 2\kappa\varepsilon. \tag{16}$$

Integral of (14) can be calculated approximately:

$$\begin{aligned} g(\omega) &= -\frac{2}{3\pi}\int \omega^3 e^{-\frac{\lambda\omega^2}{f}} r^2\left(1-\frac{r_-}{r}\right)^{\frac{2a}{1+a^2}}\frac{1}{f'}\left(-\frac{1}{f^2}\right)f' dr \\ &= -\frac{2}{3\pi}\int \omega^3 e^{-\frac{\lambda\omega^2}{f}}\frac{r^2}{f'}\left(1-\frac{r_-}{r}\right)^{\frac{2a}{1+a^2}} d\left(\frac{1}{f}\right) \\ &\approx \frac{2}{3\pi}\frac{r_+^2\omega}{\lambda f'(r_+)}\left(1-\frac{r_-}{r_+}\right)^{\frac{2a}{1+a^2}} e^{-\frac{\lambda\omega^2}{f(r_++\varepsilon)}}. \end{aligned} \tag{17}$$

The last step of (17) is valid because of $\frac{r^2}{f'}(1-\frac{r_-}{r})^{2a/(1+a^2)}$ is a slowly changed function of r at the vicinity of horizon. From (17), it is easy to verify that the number of quantum states less than energy ω is finity at the horizon. Thus we expect statistical-mechanical entropy can be obtained without any artificial cutoff.

The free energy for scalar particle is given by

$$\begin{aligned} F(\beta) &= \frac{1}{\beta}\int dg(\omega)\ln\left(1-e^{-\beta\omega}\right) \\ &= -\int_0^\infty \frac{g(\omega)}{e^{\beta\omega}-1}d\omega \\ &= -\frac{2}{3\pi}\int_0^\infty \frac{r_+^2\omega}{\lambda f'(r_+)}\left(1-\frac{r_-}{r_+}\right)^{\frac{2a}{1+a^2}} e^{-\frac{\lambda\omega^2}{f(r_++\varepsilon)}}\frac{1}{e^{\beta\omega}-1}d\omega. \end{aligned} \tag{18}$$

The entropy reads

$$\begin{aligned} S &= \beta^2\frac{\partial F}{\partial\beta} \\ &= \frac{2\beta^{-1}}{3\pi}\frac{r_+^2}{\lambda f'(r_+)}\left(1-\frac{r_-}{r_+}\right)^{\frac{2a}{1+a^2}}\int_0^\infty \frac{x^2e^x}{(e^x-1)^2}e^{-\frac{\lambda\omega^2}{\beta^2 f(r_++\varepsilon)}}d\omega. \end{aligned} \tag{19}$$

The horizon area of GHS black hole is

$$A = \int r_+^2\left(1-\frac{r_-}{r_+}\right)^{\frac{2a^2}{1+a^2}}\sin\theta d\theta d\varphi = 4\pi r_+^2\left(1-\frac{r_-}{r_+}\right)^{\frac{2a^2}{1+a^2}}. \tag{20}$$

Substituting (15)–(16) and (20) into (19), the entropy is finally given by

$$\begin{aligned} S &= \frac{A}{24\pi^3\lambda}\int_0^\infty \frac{x^2e^x}{(e^x-1)^2}e^{-\frac{x^2}{2\pi^2}}dx \\ &= \frac{C}{24\pi^3\lambda}A, \end{aligned} \tag{21}$$

where C is a constant expressed by a integral which can be calculated numerically as follows

$$C = \int_0^\infty \frac{x^2e^x}{(e^x-1)^2}e^{-\frac{x^2}{2\pi^2}}dx \simeq 2.48. \tag{22}$$

3 Conclusion

We have studied the statistical-mechanical entropy arising from the scalar field in the GHS black hole spacetime using the state density corrected by the generalized uncertainty principle. The number of quantum states with the energy less than ω is finity at horizon. Via this number of quantum states convergent entropy can be obtained without any artificial cut-off. Calculation shows that the entropy is proportional to the horizon area. The proportional coefficient contain a constant expressed by a integral which can be calculated numerically.

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